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An Analogue of Holstein-Primakoff and Dyson Realizations for Lie Superalgebras.
The Lie superalgebra sl(1/n)

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Abstract. An analogue of the Holstein-Primakoff and of the Dyson realization for the Lie superalgebra $sl(1/n)$ is written down. The expressions are the same as for the Lie algebra $sl(n+1)$, however in the latter the Bose operators have to be replaced with Fermi operators.

In the present letter we extend the concept of Holstein-Primakoff (H-P) and of Dyson (D) realizations to case of Lie superalgebras and, more precisely, to the Lie superalgebras $sl(1/n)$ for any n.

Initially the H-P and the D realizations were given for $sl(2)$ [1, 2]. It took some time the results to be generalized to the Lie algebras $gl(n)$ [3], $so(2n)$ [4] and $sp(2n)$ [5]. The extension to the case of quantum algebras is available so far only for $sl(2)$ [6] and $sl(3)$ [7]. To our best knowledge results on H-P or D realizations for Lie superalgebras were not published in the literature so far.

The motivation for the present work stems from the observation that the Holstein-Primakoff and the Dyson realizations play important role in various branches of physics. It is not possible to mention all contributions. For applications in nuclear physics see [8, 9] and the references therein; some early applications in the spin-wave theory can be found in the book of Kittel [10] (see [11] for more recent results), but there are, certainly, several other publications.

The H-P and D realization for $sl(1/n)$ we are going to present is an analogue of the one given by Okubo [3] for the Lie algebra $sl(n)$. Nevertheless it is only an analogue. The point is that we give a realization of the generators in terms of Fermi creation and annihilation operators (CAOs), whereas all H-P and D realizations are imbeddings into enveloping algebras of Bose CAOs. The similarity however is striking: replacing in the

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expressions for $sl(n+1)$ the Bose CAOs with Fermi CAOs one obtains the H-P and the D realizations for $sl(1/n)$.

We proceed to recall the definition of the Lie superalgebra (LS) $sl(1/n)$ in the notation we are going to use. Let U be the (free complex) associative unital (= with unity) algebra of the indeterminants $a_1^\pm, \ldots, a_n^\pm$ subject to the relations (bellow and throughout $[x,y] = xy - yx$, $\{x,y\} = xy + yx$)

$$[\{a_i^+, a_j^-\}, a_k^+] = \delta_{jk} a_i^+ - \delta_{ij} a_k^+, \quad [\{a_i^+, a_j^-\}, a_k^-] = -\delta_{ik} a_j^- + \delta_{ij} a_k^-, \quad \{a_i^+, a_j^+\} = \{a_i^-, a_j^-\} = 0. \qquad (1)$$

For the sake of convenience we refer to a_i^+ and to a_i^- as to creation and annihilation operators, respectively. Introduce a $\mathbb{Z}_2$−grading in U, setting

$$deg(a_i^\pm) = \bar{1}, \quad i = 1, \ldots, n. \qquad (2)$$

Then U is an (infinite-dimensional) associative superalgebra, which is also a LS with respect to the product (supercommutator) defined between every two homogeneous elements $x, y \in U$ as

$$[\![x, y]\!] = xy - (-1)^{deg(x)deg(y)} yx; \qquad (3)$$

its finite-dimensional subspace

$$A = \text{lin.env.}\{a_i^\pm, \{a_j^+, a_k^-\} | i, j, k = 1, \ldots, n\} \subset U \qquad (4)$$

is a Lie superalgebra with odd generators $a_1^\pm, \ldots, a_n^\pm$.

PROPOSITION 1. *The* LS *A is (isomorphic to) $sl(1/n)$, $A = sl(1/n)$. U is its universal enveloping algebra,* $U = U[sl(1/n)]$.

Proof. The proof can be found in [12]. We recall the main steps. Let $e_{A,B}$, $A, B = 0, 1, 2, \ldots, n$ be a $(n+1) \times (n+1)$ matrix with 1 on the intersection of the A^{th} row and B^{th} column and zero elsewhere. Then (in a lowest exact matrix representation [13]) $sl(1/n)$ is a direct space sum of its even subalgebra

$$sl(1/n)_{\bar{0}} = gl(n) = lin.env.\{e_{ij} + \delta_{ij} e_{00} | i, j = 1, \ldots, n\} \qquad (5)$$

and the odd subspace

$$sl(1/n)_{\bar{1}} = lin.env.\{e_{0i},\ e_{i0} | i = 1, \ldots, n\}. \qquad (6)$$

The one to one correspondence between the representation independent Cartan-Weyl generators (4) and the generators in the above representation read:

$$a_i^+ \leftrightarrow e_{i0}, \quad a_i^- \leftrightarrow e_{0i}, \quad \{a_i^+, a_j^-\} \leftrightarrow e_{ij} + \delta_{ij} e_{00}, \quad i, j = 1, \ldots, n. \qquad (7)$$

It is straightforward to check that the above map is a LS isomorphism. In particular only from the relations (1) (and the graded Jacoby identity) one derives also the commutation relations between all even generators

$$[\{a_i^+, a_j^-\}, \{a_k^+, a_l^-\}] = \delta_{jk} \{a_i^+, a_l^-\} - \delta_{il} \{a_k^+, a_j^-\}. \qquad (8)$$

By definition $U[sl(1/n)]$ is the associative unital algebra of $a_1^\pm, \ldots, a_n^\pm$ subject to the relations (1) and (8). Since (8) follow from (1), one can skip them. Therefore $U = U[sl(1/n)]$.

Let $f_i^\pm,\ i = 1, \ldots, n$ be n pairs of Fermi CAOs,

$$\{f_i^+, f_j^-\} = \delta_{ij}, \quad \{f_i^+, f_j^+\} = \{f_i^-, f_j^-\} = 0. \tag{9}$$

Denote by $W(0/n)$ the Fermi superalgebra, namely the 4^n-dimensional associative unital algebra with relations (9) and $\mathbb{Z}_2$ grading induced from $deg(f_i^\pm) = \bar{1}$.

Consider the following operators in $W(0/n)$

$$\varphi(a_i^+) = f_i^+, \quad \varphi(a_i^-) = (p - N)f_i^-, \quad \text{where } N = \sum_{k=1}^n f_k^+ f_k^-, \quad i = 1, \ldots, n. \tag{10}$$

It is an easy exercise to verify that the operators (10) satisfy the relations (1) and moreover that they are linearly independent. Thus, we have obtained the following result.

PROPOSITION 2. *The map φ defines an isomorphism of $sl(1/n)$ into $W(0/n)$ for any complex number p.*

This realization of $sl(1/n)$ in $W(n)$ is an analogue of the Dyson realization of $sl(n+1)$ [2]. The latter can be formally obtained form (10) replacing the Fermi operators with Bose CAOs

$$[b_i^-, b_j^+] = \delta_{ij}, \quad [b_i^+, b_j^+] = [b_i^-, b_j^-] = 0. \tag{11}$$

In that case the linear envelope of

$$b_i^+, \quad (p - N)b_i^-, \quad \text{and } [b_i^+, (p - N)b_j^-], \quad i, j = 1, \ldots, n \tag{12}$$

span a realization of $gl(n + 1)$.

The Dyson realization defines a 2^n-dimensional representation of $sl(1/n)$ in the Fermi Fock space $F(0/n)$. Let

$$|p; \Theta\rangle \equiv |p; \theta_1, \theta_2, \ldots, \theta_n\rangle = (f_1^+)^{\theta_1}(f_2^+)^{\theta_2} \ldots (f_n^+)^{\theta_n}|0\rangle, \quad \theta_1, \ldots, \theta_n = 0, 1, \tag{13}$$

be the usual orthonormed Fock basis in $F(0/n)$ and let $|p; \Theta\rangle_{\pm i}$ be a vector obtained from $|p; \Theta\rangle$ after the replacement of θ_i with $\theta_i \pm 1$. The transformations of the basis under the action of the $sl(1/n)$ generators read:

$$\varphi(a_i^+)|p; \Theta\rangle = (1 - \theta_i)(-1)^{\theta_1+\ldots+\theta_{i-1}}|p; \Theta\rangle_{+i}, \quad \varphi(a_i^-)|p; \Theta\rangle = \theta_i(-1)^{\theta_1+\ldots+\theta_{i-1}}(p - \sum_{k=1}^n \theta_k + 1)|p; \Theta\rangle_{-i}. \tag{14}$$

In the generic case the representation of $sl(1/n)$ in $F(0/n)$ is irreducible. If however $p = 1, 2, 3, \ldots, n-1$, the representation is indecomposible. Due to the factor $(p - \sum_{k=1}^n \theta_k + 1)$ the subspace of all vectors with $\sum_{k=1}^n \theta_k > p$ is an invariant subspace $F(0/n)_{inv} \subset F(0/n)$. The factor space $F(0/n)_{irr} = F(0/n)/F(0/n)_{inv}$ is irreducible. By definition such representations are called atypical [13].

The advantage of the Dyson realization (10) is its simplicity. Its disadvantage stems from the observation that the representation is not unitary, i.e., the hermitian conjugate to $\varphi(a_i^-)$ is not equal to $\varphi(a_i^+)$, i.e.,

$$(\varphi(a_i^-))^\dagger = \varphi(a_i^+) \tag{15}$$

does not hold.

It turns out all representations in $F(0/n)_{irr}$ corresponding to integer nonnegative p, $\quad p \in \mathbf{Z}_+$, are equivalent to unitary representations. To this end one has to change in an appropriate way the scalar product in $F(0/n)_{irr}$. We give the final results.

PROPOSITION 3. *The map π defined as*

$$\pi(a_i^+) = f_i^+ \sqrt{p-N}, \quad \pi(a_i^-) = \sqrt{p-N}\, f_i^- \tag{16}$$

is an isomorphism of $sl(1/n)$ into $W(0/n)$ for any $p \in \mathbf{Z}_+$. The space $F(0/n)_{irr}$ with an orthonormed basis consisting of all vectors

$$|p;\Theta\rangle \equiv |p;\theta_1,\theta_2,\ldots,\theta_n\rangle, \quad \textit{such that} \quad \sum_{k=1}^n \theta_k \le p, \quad \theta_1,\ldots,\theta_n = 0,1, \tag{17}$$

carries an unitary irreducible representation of $sl(1/n)$, i.e., (15) holds. The transformation of the basis reads:

$$\pi(a_i^+)|p;\Theta\rangle = (1-\theta_i)(-1)^{\theta_1+\ldots+\theta_{i-1}}\sqrt{p-\sum_{k=1}^n \theta_k}\ |p;\Theta\rangle_{+i}, \tag{18a}$$

$$\pi(a_i^-)|p;\Theta\rangle = \theta_i(-1)^{\theta_1+\ldots+\theta_{i-1}}\sqrt{p-\sum_{k=1}^n \theta_k+1}\ |p;\Theta\rangle_{-i}. \tag{18b}$$

If $p<n$ the representation is atypical.

The supercommutation relations (1) can be checked within the particular representation, acting on the basis vectors (17) according to (18). One can proceed in a more algebraic way first expanding $\sqrt{p-N}$ in a series with respect to N, so that it is explicit that $\sqrt{p-N} \in W(0/n)$.

The expressions (16) can be considered as analogue of the Holstein-Primakoff realization for the Lie superalgebra $sl(1/n)$. Also here, replacing the fermions in (16) with bosons one obtains the H-P realizations for sl(n+1) as given by Okubo [3].

We wrote down explicit expressions in terms of fermions only for the $\pi(a_i^\pm)$. The rest of the Cartan-Weyl generators, namely $\pi(\{a_i^+,a_j^-\}) = \{\pi(a_i^+),\pi(a_j^-)\}$ follow from (16). One can express also the unit operator in $F(0/n)_{irr}$ in terms of fermions, thus extending the results to the LS gl(1/n). The more detailed results will be given elsewhere directly for the LS gl(m/n).

Let us note that the explicit expressions for all finite-dimensional irreducible representations of $sl(1/n)$ and $gl(1/n)$ are known [14]. The formulas are however extremely involved. Here we have obtained a small

subset of all representations, which description is however simple and is realized in familiar for physics Fock spaces.

The above construction holds also for the case $n = \infty$. In that case the representation space of $sl(1/\infty)$ has an orthonormed basis, consisting of all vectors, such that $\sum_{k=1}^{\infty} \theta_k \leq p$. All representations are atypical infinite-dimension irreducible representations.

One could expect that the above results may help to give H-P and D realizations for the quantum algebra $U_q[sl(1/n)]$. The explicit representations of the Chevalley generators in a Gelfand-Zetlin basis are known [15]. As a first step here one has to pass to the corresponding deformed CAOs, expressing them in terms of the Chevalley operators, and write subsequently the transformation of the basis under the action of these operators.

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